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Weyl algebras and knots

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Abstract

In this paper we put forward results on the invariant $\mathcal{F}$-module of a virtual knot investigated by the first named author where $\mathcal{F}$ is the algebra with two invertible generators A, B and one relation $A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A$. For flat knots and links the two sides of the relation equation are put equal to unity and the algebra becomes the Weyl algebra. If this is perturbed and the two sides of the relation equation are put equal to a general element, q, of the ground ring, then the resulting module lays claim to be the correct generalization of the Alexander module. Many finite dimensional representations are given together with calculations.

Keywords: Virtual and flat links; Quantum Weyl algebras

1. Introduction

The first named author [7] and others recently introduced a general method associating a module with an arbitrary oriented link diagram on an oriented surface. The module is generated by the arcs obtained by splitting the diagram at its crossings. The relations are associated with the crossings and depend on a choice of two invertible elements A, B of an associative algebra such that

$$A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A.$$

This, *fundamental relation*, ensures that the module is invariant under the Reidemeister moves on the diagram and provides thus an invariant of oriented links in surfaces crossed with an interval.

This construction generalizes the classical Alexander module of a link in S^3 as well as its natural extensions to surface links. Note however that the classical construction uses commuting A, B which conceals non-commutative ramifications.

In the present paper we show that the fundamental equation has a natural family of solutions arising from the so-called quantum Weyl algebras. For physicists this is a quantum version of the Heisenberg algebra of a harmonic oscillator. Given an element q of a (commutative) ground ring K, we define the *quantum Weyl algebra* W_q to be the

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K-algebra generated by four elements u, u^{-1}, v, v^{-1} subject to the relations $uu^{-1} = u^{-1}u = vv^{-1} = v^{-1}v = 1$ and $uv - qvu = 1$.

We show that $A = v^{-1}u^{-1} \in W_q$ and $B = u \in W_q$ satisfy the fundamental relation. Applying Fenn's method to these A, B, we associate with every link diagram D a W_q-module $\mathcal{M}(D)$. It has a square presentation matrix so that one is tempted to take the determinant (or subdeterminants) as in the Alexander theory. However, the algebra W_q is non-commutative and the determinants do not make sense. One solution is to plug in matrix representations of W_q over K and take the determinants of the resulting matrices; see for example, [1]. This gives interesting link invariants satisfying the same skein relation as the Alexander–Conway polynomial of links in S^3 and dependent on the choice of a matrix representation of W_q. As expected, these invariants are extensions of the known Alexander invariants of links in S^3. For examples involving quaternions see [3] and [4]. A paper by Fenn involving generalised quaternions is in preparation. A good reference for the algebra involved is [12].

Parallel constructions work for closed curves on oriented surfaces and produce homotopy invariants of such curves. Here we involve an extension of the classical Weyl algebra, obtained by putting $q = 1$ in W_q.

The plan of the paper is as follows. In Sections 2 and 3 we discuss the extension of classical Weyl algebra and use it to produce invariants of closed curves on surfaces. In Sections 4 and 5 we discuss the quantum Weyl algebra and use it to produce link invariants. In Section 6 we give proofs of several claims made in Sections 3 and 5. In Section 7 we consider representations of the virtual braid groups arising from quantum Weyl algebras. In Section 8 some calculations are given. One of the examples shows that the shadow of the Kishino knot is non-trivial, verifying a result from [11].

2. The Weyl algebra

In this section we consider the Weyl algebra and variants. A good reference for the details of this section is Cohn's book [5].

Let K be a commutative ring and W^0 be the K-algebra generated by u, v and with the relation

$$uv - vu = 1.$$

Then W^0 is called the *Weyl algebra* on u, v over K. If K is a field of characteristic 0 then W^0 is a simple ring, that is all two-sided ideals are trivial; see [5], pp. 362–363.

We now extend W^0 so that u, v are invertible. Let W be the *extended Weyl algebra* defined as the quotient of the K-algebra generated by $u^{\pm 1}, v^{\pm 1}$ by the ideal generated by $uv - vu - 1$. Although we shall not formally need it, note that the natural algebra homomorphism $W^0 \to W$, sending u to u and v to v, is injective. In Section 4 we will give a proof of a more general statement.

Let $M_n(K)$ denote the algebra of $n \times n$ matrices with entries in K. We would like to represent W by matrices in $M_n(K)$. Note that, given v, the relation $uv - vu = 1$ is affine in u and so any solution can be written $u = u_P + u_H$ where u_P is any particular solution and u_H commutes with v, say a polynomial in v.

Theorem 2.1. *The extended Weyl algebra W over a field of characteristic 0, K, has no non-trivial representations in $M_n(K)$.*

Proof. Since W^0 is simple, i.e., all two-sided ideals are trivial, any representation of W^0 is either trivial or faithful. But W^0 has infinite dimension over K and so any finite dimensional representation collapses on W^0 and hence on W. □

An obvious question is whether W is simple over a field of characteristic 0. Note that W is not simple if the field has characteristic a prime p since $K[u^p, v^p]$ is central.

Traditionally W acts on the algebra $C^\infty(\mathbf{C} - \{0\})$ by

$$u(f) = f' + f, \qquad v(f) = xf$$

where $f \in C^\infty(\mathbf{C} - \{0\})$.

For a finite dimensional representation, consider the truncated polynomial ring

$$R = K[x]/(x^n = 0)$$

where K is a field of characteristic dividing n. Let $I = i_0 + i_1 x + \cdots$ and $J = j_0 + j_1 x + \cdots$ be elements of R. If $i_0 \neq 0$ and $i_1 \neq 0$ then I and its derivative I' are units. In fact $(I')^{-1} = k_0 + k_1 x + k_2 x^2 + \cdots$ where

$$k_0 = i_1^{-1}, \qquad k_1 = -2i_2 i_1^{-2}, \qquad k_2 = (4i_2^2 - 3i_1 i_3)i_1^{-3}, \ldots .$$

Further coefficients can be determined from the difference equation

$$i_1 k_r + 2i_2 k_{r-1} + \cdots + (r+1)i_{r+1}k_0 = 0.$$

Define the K-linear operators $u, v : R \to R$ by

$$u(f) = \frac{f'}{I'} + Jf, \qquad v(f) = If.$$

Then it is easily seen that v is invertible and $uv - vu = 1$. The matrices of u, v with respect to the basis $\{1, x, x^2, \ldots, x^{n-1}\}$ are

$$u = \begin{pmatrix} j_0 & j_1 & j_2 & \ldots & j_{n-2} & j_{n-1} \\ k_0 & j_0 + k_1 & j_1 + k_2 & \ldots & j_{n-3} + k_{n-2} & j_{n-2} + k_{n-1} \\ 0 & 2k_0 & j_0 + 2k_1 & \ldots & j_{n-4} + 2k_{n-3} & j_{n-3} + 2k_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & (n-1)k_0 & j_0 + (n-1)k_1 \end{pmatrix}$$

$$v = \begin{pmatrix} i_0 & i_1 & \ldots & i_{n-1} \\ 0 & i_0 & \ldots & i_{n-2} \\ 0 & 0 & \ldots & i_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & i_0 \end{pmatrix}.$$

Then u is invertible provided its determinant is non-zero. This imposes a polynomial condition on the variables, $i_0, i_1, \ldots, j_0, j_1, \ldots$.

If K has characteristic p dividing n, then

$$u = \begin{pmatrix} x & a_1 & 0 & \ldots & 0 \\ 0 & x & a_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_{n-1} \\ 0 & 0 & 0 & \ldots & x \end{pmatrix} \qquad v = \begin{pmatrix} y & 0 & \ldots & 0 & 0 \\ 1/a_1 & y & \ldots & 0 & 0 \\ 0 & 2/a_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & (n-1)/a_{n-1} & y \end{pmatrix}$$

is a non-trivial representation of W in $M_n(K)$, with $n+1$ parameters $x, y, a_1, \ldots, a_{n-1} \in K$ where $x \neq 0$, $y \neq 0$.

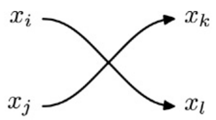
3. W-modules and flat links

By a *system of loops on a surface*, we mean a mapping from a disjoint union of a finite number of oriented circles into a compact oriented surface with empty boundary. Two systems of loops on surfaces are *stably equivalent* if they can be related by a finite sequence of the following operations: (i) homotopy of loops on the surface; (ii) composing with an orientation preserving homeomorphism of surfaces; (iii) attaching a 1-handle to the ambient surface away from the loops or removing such a handle.

We call the stable equivalence classes of systems of loops on surfaces *flat links*. When there is only one loop, we can speak of *flat knots*. Elsewhere they are called flat virtuals [13] and virtual strings [17].

We now define a W-module, for each flat link L. Represent L by a system of loops on a surface lying in general position. This system has a finite number, n, of intersection points, where the loops cross (or self-cross) transversely. Let m be the number of loops in the system having no self-intersection points and missing the other loops. The *arcs* are the $N = 2n + m$ components of our system of loops with the intersection points removed. Note that arcs do not pass through crossing points. Label the arcs $x_1, \ldots, x_N$ in an arbitrary way. These labels will be the generators of the module.

Pick an associative K-algebra with unit R and fix elements $A, B, C, D \in R$. Each self-intersection point contributes two linear relations between the symbols $x_1, \ldots, x_N$ as indicated by the following diagram where the input arcs arriving at the crossing are labelled x_i, x_j, the output arcs leaving the crossing are labelled x_k, x_l and we assume that the surface is oriented counterclockwise.



$$x_l = Ax_j + Bx_i, \ x_k = Cx_j + Dx_i$$

Quotienting the free R-module with generators $x_1, \ldots, x_N$ by the $2n$ relations derived from the crossings, we obtain an R-module denoted as $\mathcal{M}_{A,B,C,D}(L)$.

Theorem 3.2. *Let $R = W$ and $A = v^{-1}u^{-1}$, $B = u$, $C = uvu^{-1}v^{-1}u^{-1}vuv^{-1}u^{-1}$, $D = -u^{-1}v^{-1}$ (where $uv - vu = 1$). Then the W-module $\mathcal{M}_{A,B,C,D}(L)$ is a stable equivalence invariant of L.*

This theorem will be proven in Section 6.

Let $\mathcal{M}(L)$ denote the W-module defined in Theorem 3.2. Workable invariants can be derived from this module as follows. If $\mathcal{M}'$ is any W-module then $\mathrm{Hom}_W(\mathcal{M}(L), \mathcal{M}')$ is a finitely generated K-module. If K is a field, we can set

$$d_{\mathcal{M}'}(L) = \dim_K \mathrm{Hom}_W(\mathcal{M}(L), \mathcal{M}').$$

Then $d_{\mathcal{M}'}(L)$ is an integer invariant of L. For the trivial m-component flat link, $\mathcal{M}(L) = W^m$ and $d_{\mathcal{M}'} = (\dim_K(\mathcal{M}'))^m$.

A presentation of the module $\mathcal{M}(L)$ as above is determined by an $N \times N$ matrix M with entries in W (actually the entries are $A, B, C, D, 0, -1$). Any matrix representation $\rho : W \to M_k(K)$ transforms M into a $2Nk \times 2Nk$ square matrix. Its determinant Δ_0 is an invariant of L up to multiplication by powers of the determinant of $\rho(B)$; see [7]. We can also consider the ideal I_r in K generated by the codimension r subdeterminants for integer $r > 0$. For suitable K, this has a greatest common divisor $\Delta_r(L)$ which is an invariant of L up to multiplication by units.

4. The quantum Weyl algebra

Let W_q^0 be the algebra over the commutative ring K generated by u, v and with relation $uv - qvu = 1$ where q is an invertible element of K. This algebra is called the *q-oscillator algebra* on u, v in [6]. The variable h used in [6] is equal to 1 here.

We define the *quantum Weyl algebra* W_q to be the quotient of the K-algebra in $u^{\pm 1}, v^{\pm 1}$, by the ideal generated by $uv - qvu - 1$.

Lemma 4.3. *The algebra homomorphism $\alpha : W_q^0 \to W_q$, sending u to u and v to v, is injective.*

Proof. Recall first the definition of the ring of non-commuting polynomials, $R[x; \sigma]$, where R is a ring and σ is a ring automorphism of R. The ring $R[x; \sigma]$ is obtained from the free algebra on R with an added generator x by imposing the condition $xr = \sigma(r)x$ for all $r \in R$. Any element of $R[x; \sigma]$ can be written uniquely as a "finite polynomial" in x, namely $r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n$ where all r_i lie in R. Similarly, we define the ring of non-commuting Laurent polynomials, $R[x^{\pm 1}; \sigma]$, where any element is a Laurent polynomial $r_{-m}x^{-m} + \cdots + r_n x^n$. Of course if σ is the identity, then these definitions give the usual ring of polynomials, $R[x]$, and the ring of Laurent polynomials, $R[x^{\pm 1}]$.

Let $R = K(h)$ be the field of rational functions on one variable h with coefficients in K. In other words, R is the field of fractions of the commutative ring of polynomials $K[h]$. Let σ be the ring automorphism of R sending h to $q^{-1}(h - 1)$. Thus, σ sends an arbitrary rational function $f(h) \in R$ to $f(q^{-1}(h - 1)) \in R$. Consider the ring of non-commuting Laurent polynomials, $U = R[x^{\pm 1}; \sigma]$. In U we have the equalities

$$(hx^{-1})x - qx(hx^{-1}) = h - q\sigma(h)xx^{-1} = h - (h - 1) = 1.$$

This implies the existence of a homomorphism of K-algebras $\beta : W_q^0 \to U$ such that $\beta(u) = hx^{-1}$ and $\beta(v) = x$. The same equalities and the fact that hx^{-1} and x are invertible in U imply that there is a homomorphism of K-algebras

$\gamma : W_q \to U$ sending u to hx^{-1} and v to x. It is clear that $\beta = \gamma\alpha$. It is easy to see that β is injective. Indeed, if a polynomial $\sum_{m,n\geq 0} k_{m,n} u^m v^n$ with $k_{m,n} \in K$ lies in the kernel of β, then $\sum_{m,n\geq 0} k_{m,n} h^m x^{n-m} = 0$ in U. For any integer s, the monomial x^s appears here with coefficient $\sum_{m\geq 0} k_{m,m+s} h^m$ which therefore must be 0 in $R = K(h)$. Hence $k_{m,n}$ are all equal to 0. The injectivity of β and the equality $\beta = \gamma\alpha$ imply that α is injective. □

Of course if $q = 1$ then W_q^0 and W_q reduce to W^0 and W considered earlier and the above lemma encompasses them also. From now on we shall assume that $1 - q$ is invertible.

We now study finite dimensional representations of W_q.

Lemma 4.4. *Let u, v be invertible $n \times n$ matrices over K satisfying $uv - qvu = 1$. If the linear map $K^n \to K^n$ defined by $x \mapsto vx$ has no invariant subspaces other than 0 and K^n, then either $u = (1-q)^{-1}v^{-1}$ or q is an n-th root of unity.*

Proof. Set

$$u = u_H + \frac{1}{1-q} v^{-1}.$$

Then u is defined by u_H and conversely. Moreover u_H q-commutes with v,

$$u_H v = q v u_H.$$

Let X be the kernel of the linear map $K^n \to K^n$ defined by $x \mapsto u_H x$. Then X is an invariant subspace of v and so is either K^n or $\{0\}$. In the first case $u = (1-q)^{-1}v^{-1}$. In the second case take determinants. This shows that $q^n = 1$. □

In view of this lemma, we will restrict our attention to triangular matrices. Suppose that u, v are upper triangular with diagonal elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively. Then it is easy to see that the b's are related to the a's by

$$a_i b_i = \frac{1}{1-q}.$$

Moreover if $a_i \neq q a_{i+1}, b_{i+1} \neq q b_i$ for all i then u, v commute. An example where this does not happen is

$$u = \begin{pmatrix} q^{n-1}a & b^{n-2}d & 0 & \ldots & 0 \\ 0 & q^{n-2}a & b^{n-3}d & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & d \\ 0 & 0 & 0 & \ldots & a \end{pmatrix}, \qquad v = \begin{pmatrix} c & e & 0 & \ldots & 0 \\ 0 & qc & qb^{-1}e & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & (qb^{-1})^{n-2}e \\ 0 & 0 & 0 & \ldots & q^{n-1}c \end{pmatrix}$$

where $c = 1/(aq^{n-1}(1-q))$.

Let us look for representations of the form

$$u = \begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 \\ b_1 & a_2 & 0 & \ldots & 0 \\ 0 & b_2 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}, \qquad v = \begin{pmatrix} c_1 & d_1 & 0 & \ldots & 0 \\ 0 & c_2 & d_2 & \ldots & 0 \\ 0 & 0 & c_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & c_n \end{pmatrix}.$$

These will satisfy $uv - qvu = 1$ provided $a_i = q^{n-i}a$, $c_i = q^{n-i}c$ for some a, c and

$$b_i d_i = q^{n-2i} + q^{n-2i+1} + \cdots + q^{n-i-1} - q^{-i+1} - q^{-i+2} - \cdots - q^{-1}, \quad i = 1, 2, \ldots, n-1$$

giving a representation of W_q with $n + 1$ parameters.

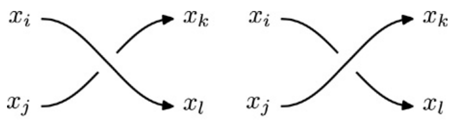
5. Virtual links

We will consider links in 3-manifolds obtained by thickening of a compact oriented surface, Σ, with empty boundary. These links are represented by smooth embeddings of a disjoint union of a finite number of oriented circles into $\Sigma \times [0, 1]$. Under the projection $\Sigma \times [0, 1] \to \Sigma$ a link defines a link diagram on Σ which we will assume in general position with only transverse double points. The over and under arcs can be distinguished in the usual way. This establishes a bijection between ambient isotopy classes of links in $\Sigma \times [0, 1]$ and link diagrams on Σ modulo the Reidemeister moves.

Two links are said to be *stably equivalent* if they can be related by a finite sequence of the following operations: (i) Reidemeister moves on a link diagram on the surface; (ii) transferring a diagram with an orientation preserving homeomorphism of surfaces; (iii) attaching a 1-handle to the ambient surface away from the diagram or removing such a handle. The stable equivalence class of a link L is a *surface link* or a *virtual link*; see [13,14]. We will shorten this to just link. There is a natural map from links to flat links defined by forgetting the over and under information on a link diagram.

In generalization of the method used for flat links we define a W_q-module, $\mathcal{M}_q(L)$, for each link L. It is called the *quantum Weyl* module of L and is defined by generators and relations using a diagram of L. Suppose that the diagram has n double points and m simple closed loops disjoint from other loops. The *arcs* are the $N = 2n + m$ components of the diagram with the intersection points removed. Let the arcs be labelled $x_1, \ldots, x_N$. These labels will be the generators of the module.

Pick an associative K-algebra with unit R and fix elements $A, B, C, D \in R$. Each self-intersection point contributes two relations as indicated by the following diagram.



$$x_l = Ax_j + Bx_i \qquad x_j = Ax_l + Bx_k$$
$$x_k = Cx_j + Dx_i \qquad x_i = Cx_l + Dx_k$$

We get an R-module as we did for flat links which again we denote by $\mathcal{M}_{A,B,C,D}(L)$.

Theorem 5.5. *Let $R = W_q$ and $A = v^{-1}u^{-1}$, $B = u$, $C = quvu^{-1}v^{-1}u^{-1}v^{-1}u^{-1}vu$, $D = 1 - q - u^{-1}v^{-1}$ (where $uv - qvu = 1$). Then the W_q-module $\mathcal{M}_{A,B,C,D}(L)$ is a stable equivalence invariant of L.*

The proof will be given in Section 6. As in the case of flat links we may define rank invariants $d_{\mathcal{M}'}(L)$, ideal invariants $I_r^{(q)}$ and determinantal invariants Δ_0 and Δ_r.

6. Proof of Theorems 3.2 and 5.5

We say that elements A, B of an associative K-algebra with unit R satisfy the *fundamental* relation if they are invertible and

$$A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A. \tag{1}$$

Recall the following theorem of Fenn [7,3,4].

Theorem 6.6. *If $A, B \in R$ satisfy the fundamental relation, $C = A^{-1}B^{-1}A(1 - A)$ is invertible in R, and $D = 1 - A^{-1}B^{-1}AB$, then for any link L, the module $\mathcal{M}_{A,B,C,D}(L)$ is a stable equivalence invariant of L.*

To deduce Theorem 5.5 from Theorem 6.6 we need the following lemma.

Lemma 6.7. *If $q \in K$ and A, B are invertible elements of R satisfying the relation*

$$B = BA^{-1} - qA^{-1}B, \tag{2}$$

then A, B satisfy the fundamental equation (1).

Proof. We have

$$qB^{-1} = A^{-1}B^{-1}A - B^{-1}A$$

as can be seen by multiplying (2) on the right by $B^{-1}A$ and then multiplying on the left by B^{-1}. Then the requirement of the fundamental equation is that qB^{-1} commutes with B which holds since q is an element of the ground ring. □

We apply this lemma to $R = W_q$ and $A = v^{-1}u^{-1}$, $B = u$. It is easy to check that A, B satisfy (2) and therefore A, B satisfy the fundamental relation. We have

$$C = A^{-1}B^{-1}A(1 - A) = uvu^{-1}v^{-1}u^{-1}(1 - v^{-1}u^{-1}) = quvu^{-1}v^{-1}u^{-1}v^{-1}u^{-1}vu$$

where we use the equality $1 - v^{-1}u^{-1} = qv^{-1}u^{-1}vu$ obtained from $uv - qvu = 1$ via multiplication by $v^{-1}u^{-1}$. Note that C is invertible in W_q. Similarly,

$$D = 1 - A^{-1}B^{-1}AB = 1 - uvu^{-1}v^{-1} = 1 - q - u^{-1}v^{-1}.$$

Therefore Theorem 5.5 follows from Theorem 6.6.

To prove Theorem 3.2 it is enough to substitute $q = 1$ in Theorem 5.5 and to observe that for our choice of $A, B, C, D \in W$, the matrix

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(W)$$

is equal to its inverse: $S = S^{-1}$. The latter is a direct consequence of Claim (c) of the following lemma (Claims (a) and (b) of this lemma will be used in the next section).

Lemma 6.8. *Suppose $A, B \in R$ satisfy (1) and C, D are defined by*

$$C = A^{-1}B^{-1}A(1 - A), \qquad D = 1 - A^{-1}B^{-1}AB.$$

(a) *Let*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(R).$$

Then S is invertible if and only if C is invertible.

(b) *Put*

$$S_1 = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_3(R), \qquad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix} \in M_3(R).$$

Then

$$S_1 S_2 S_1 = S_2 S_1 S_2. \tag{3}$$

(c) *Suppose that $q = (1 - A)A^{-1}B^{-1}AB$ is an element of the ground ring. Then $S^2 = (1 - q)S + q$.*

Proof. We can write S as a product of elementary matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - A^{-1} \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}.$$

If C is invertible then so is $1 - A$ and hence each matrix in the product is invertible. This proves (a).

The proof of (b) follows by basic manipulations.

To prove (c) observe that conjugating the equality $q = (1 - A)A^{-1}B^{-1}AB$ by B we obtain $q = BA^{-1}B^{-1}A - A$. Multiplying on the right by A^{-1} we obtain that $BA^{-1}B^{-1} = 1 + qA^{-1}$. The latter formula will be used in the computation of

$$S^2 = \begin{pmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{pmatrix}.$$

Substituting the values of C, D we get

$$\begin{aligned} A^2 + BC &= A^2 + (BA^{-1}B^{-1})A(1-A) \\ &= q + (1-q)A, \\ AB + BD &= AB + B - (BA^{-1}B^{-1})AB \\ &= (1-q)B, \\ CA + DC &= A^{-1}B^{-1}A^2(1-A) + (1 - A^{-1}B^{-1}AB)A^{-1}B^{-1}A(1-A) \\ &= A^{-1}B^{-1}A(1 - A^2 - (BA^{-1}B^{-1})A(1-A)) \\ &= A^{-1}B^{-1}A(1-q)(1-A) \\ &= (1-q)C, \\ CB + D^2 &= A^{-1}B^{-1}A(B - AB + (BA^{-1}B^{-1})AB) + 2D - 1 \\ &= A^{-1}B^{-1}A(1+q)B + 2D - 1 \\ &= (1+q)A^{-1}B^{-1}AB + 2D - 1 \\ &= q + (1-q)D. \end{aligned}$$

As an aid to the reader we have put brackets where the substitution $BA^{-1}B^{-1} = 1 + qA^{-1}$ takes place. □

Note: the quadratic equation in S given by (c) implies that we have a representation of the Hecke algebra, H_n, for each n, [10]. The equation can also be written as

$$q^{-1/2}S - q^{1/2}S^{-1} = q^{-1/2} - q^{1/2}.$$

7. Representations of the braid group

In this section we look at some representations of the braid group, B_n, an extension of the braid group, VB_n, and a quotient of this extension, FB_n. These representations are defined by the work in the previous section. In the case of the braid group all the representations are equivalent to the Burau representation although this is certainly not the case for the two other groups.

Let n be a positive integer. The *braid group* B_n has generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| > 1$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad i = 1, \ldots, n-1.$$

The *virtual braid group*, VB_n, is an extension of B_n with new generators $\tau_1, \tau_2, \ldots, \tau_{n-1}$ and two sorts of extra relations.

Permutation group relations:

$${\tau_i}^2 = 1$$
$$\tau_i\tau_j = \tau_j\tau_i \quad |i-j| > 1$$
$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}.$$

Mixed relations:

$$\sigma_i\tau_j = \tau_j\sigma_i \quad |i-j| > 1$$
$$\sigma_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\sigma_{i+1} \quad i = 1, \ldots, n-1.$$

The *flat braid group*, FB_n, is the quotient of VB_n by the relations $\sigma_i^2 = 1$ for all i.

Let $S : X^2 \to X^2$ be a permutation of the cartesian square of a set X. In [8] such an S is called a *switch* if

$$(S \times id)(id \times S)(S \times id) = (id \times S)(S \times id)(id \times S).$$

Examples of switches are the identity and the *twist*, T, defined by $T(a, b) = (b, a)$.

A binary operation, $(a, b) \to a^b$, is called *invertible on the right* if there exists another binary operation, $(a, b) \to a^{b^{-1}}$ such that

$$a^{bb^{-1}} = a^{b^{-1}b} = a$$

is always true. For example racks or quandles, see [9], are invertible.

A switch, S, defines two binary operations by the formula

$$S(a, b) = (b_a, a^b).$$

A switch is called a *biquandle* if both operations are invertible and

$$a^{a^{-1}} = a_{a^{a^{-1}}}$$

for all $a \in X$. Note that the original definition included the extra condition

$$a_{a^{-1}} = a^{a_{a^{-1}}}$$

but this has been shown to be unnecessary by [15].

Theorem 7.9. *A linear switch defined by a* 2×2 *matrix is a biquandle.*

Proof. See [8]. □

If $S : X^2 \to X^2$ is a switch and n is a positive integer, define the permutation, S_i, of X^n by

$$S_i = id^{i-1} \times S \times id^{n-i-1}$$

where $id : X \to X$ is the identity. Then S defines a homomorphism $r_S : B_n \to P_n(X)$ where $P_n(X)$ is the group of permutations of X^n.

Let $b(t) = \begin{pmatrix} 0 & 1 \\ t & 1-t \end{pmatrix}$ denote the Burau matrix with parameter t. Then $b(t)$ is a switch.

Theorem 7.10. *If* $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is any linear switch, then the homomorphism* r_S *is equivalent to* $r_{b(t)}$ *where* $t = (1 - A)(1 - D)$.

Proof. See [7]. □

We now extend the representation r_S to VB_n by sending the generator τ_i to T_i, where T is the twist. We will continue to call the resulting representation r_S.

Any link L is the closure of a virtual braid $\beta \in VB_n$ for some n. Let $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a linear switch where A, B, C, D are elements of an algebra R. Let $\mathcal{M}_S(L)$ denote the left R-module with $2n \times 2n$ presentation matrix $r_S(\beta) - 1$.

For example, suppose that $S = \begin{pmatrix} 1 - BC & B \\ C & 0 \end{pmatrix}$. Then the module $\mathcal{M}_S(L)$ is the Sawollek module of L with a change of variable; see [16]. This becomes the Alexander module of L if $C = 1$.

Theorem 7.11. *For any linear switch S, the module $\mathcal{M}_S(L)$ is a stable equivalence invariant of L. If the link L is classical then $\mathcal{M}_S(L)$ is equivalent to the Alexander module of L. So in particular $\Delta_0 = 0$ and Δ_1 is the Alexander polynomial with variable $t = (1 - A)(1 - D)$.*

Proof. For the invariance of the module see [7] or [8]. For classical links the braid β can be chosen in B_n and then the representation is equivalent to the Burau representation and this defines the Alexander module. □

More generally we have the following:

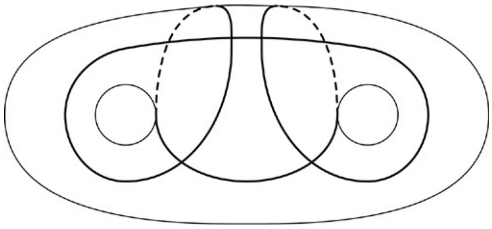
Theorem 7.12. *Let $\mathcal{M}_q(L)$ be the quantum Weyl module of a link L. Then if the generators u, v of the Weyl algebra commute, the module becomes the Sawollek module $\mathcal{M}_S(L)$ for* $S = \begin{pmatrix} 1 - q & u \\ q/u & 0 \end{pmatrix}$.

Suppose that S is a switch which satisfies $S^2 = 1$. Then there is a representation w_S of the flat braid group given by $w_S(\rho_i) = S_i$ and $w_S(\tau_i) = T_i$. However we can finesse this definition by putting $w_S(\tau_i) = T'_i$ where T' is any switch which satisfies $T'^2 = 1$ and $T_1 T_2 S_1 = S_2 T_1 T_2$. The result will now depend more heavily on the passage of the representative loop around handles.

8. Worked examples

In this section we consider various examples and work out their invariants. We are very grateful to Andrew Bartholomew who has developed the software to do the calculations. This can be freely obtained from [2].

The first example is the projection of the Kishino knot considered in [8]. If we can show that this flat knot is non-trivial then all possible lifts as virtual knots will *a fortiori* be non-trivial.



The Kishino Flat Knot

This is the closure of the braid, $k = \tau_2\sigma_1\sigma_2\sigma_1\tau_2\sigma_1\sigma_2\sigma_1$. Using the representation of the Weyl algebra given by

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad v = \begin{pmatrix} y & 0 & 0 \\ 1 & y & 0 \\ 0 & 2 & y \end{pmatrix}$$

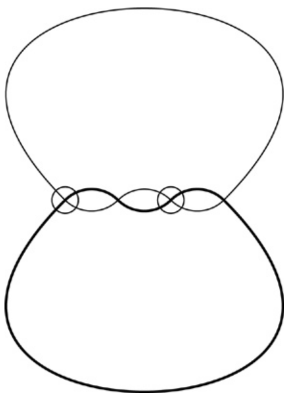
with underlying ring $\mathbb{Z}_3[y]$ and using the software developed by Bartholomew we find

$$\Delta_0 = 0, \qquad \Delta_1 = 2 + 2y.$$

Since $\Delta_1 \neq 1$, the Kishino flat knot is indeed non-trivial.

The second example consists of all possible flat knots which are the closures of braids in the flat braid group FB_2. Clearly we need only consider closures of $r_n = \tau_1\sigma_1\tau_1\sigma_1 \cdots \tau_1\sigma_1$ where $2n$ is the number of multiplicands. Let L_n denote this closure.

For example L_2, the closure of $r_2 = \tau_1\sigma_1\tau_1\sigma_1$ is illustrated below.



We will use the representation of the Weyl algebra given by

$$u = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} \qquad v = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

with underlying ring $\mathbb{Z}_2[x]$.

Theorem 8.13. *With the above representation the invariant of L_n is $\Delta_0 = x^{2n} + x^{-2n}$.*

Proof. We will look for the eigenvalues of ST. This leads to the equations

$$B\mathbf{x} + A\mathbf{y} = \lambda\mathbf{x} \qquad D\mathbf{x} + C\mathbf{y} = \lambda\mathbf{y}$$

where

$$A = \begin{pmatrix} 1/x & 1/x^2 \\ 1/x & (1+x)/x^2 \end{pmatrix} \qquad B = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$$
$$C = \begin{pmatrix} (1+x+x^2+x^4)/x^5 & (1+x^2+x^4)/x^6 \\ 1/x^4 & (1+x+x^2+x^4)/x^5 \end{pmatrix} \qquad D = \begin{pmatrix} (1+x)/x^2 & 1/x^2 \\ 1/x & 1/x \end{pmatrix}.$$

Eliminating $\mathbf{y} = A^{-1}(\lambda - B)\mathbf{x}$ we see that the matrix

$$D + (C - \lambda)A^{-1}(\lambda - B)$$

is singular. Taking determinants we get the following equation in λ:

$$\lambda^4 + \lambda^2(x^2 + x^{-2}) + 1 = 0.$$

The roots are $\lambda = x$ (twice) and $\lambda = x^{-1}$ (twice). By working over an algebraically closed field extension if necessary we can write

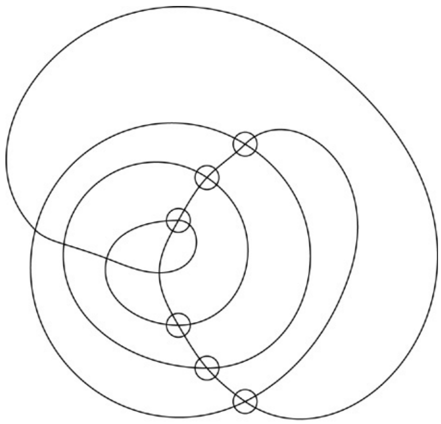
$$ST = P^{-1}UP$$

where U is upper triangular and has x, x, x^{-1}, x^{-1} down the diagonal. Then $(ST)^n = P^{-1}U^nP$ where U^n is upper triangular and has x^n, x^n, x^{-n}, x^{-n} down the diagonal. So $\Delta_0 = \det((ST)^n - 1) = x^{2n} + x^{-2n}$ as required. □

The third example we call the *whorl*, W_n. It is the closure of the flat braid

$$w_n = \tau_1\tau_2\cdots\tau_n\tau_{n-1}\cdots\tau_2\sigma_1\sigma_2\cdots\sigma_n.$$

The whorl has genus 2.



The Whorl, W_4

The values of Δ_0 for $n = 3, 4$ are

$$\Delta_0 = (x^{10} + x^4 + x^2 + 1)/x^{10}, \quad (x^{14} + x^{10} + x^8 + x^6 + x^2 + 1)/x^{12},$$

respectively. We have been unable to find a general pattern.

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